A statistical model of experimental test on Bell's inequality

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Abstract. It is demonstrated that experiments testing Bell's theorem contain a 2-level statistical structure. Bell's inequality can be derived for statistical mean value with an assumption of independence among the four sets of measurements. However, in such case, the upper bound for the CHSH-Bell observable is 4 for any and all individual experimental outcome, regardless of the locality consideration. Requirement of fair sampling should take all individual outcome into account. And it is proposed that hypothesis testing should be used to analyze the experimental data.

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Bell's inequality [1] has been the foundation of quantum entanglement and set a criterion for differentiating Quantum theory from classical theory. One of the widely used variant of the inequality was introduced by Clauser, Horne, Shimony, and Holt[2].

$$\begin{aligned} &S(a,a',b,b') \\ &= |E(a,b)+E(a,b')| + |E(a',b)-E(a',b')| < 2 \end{aligned} \tag{1}$$

with a, a', b, b' being parameters of detection apparatus.

Since [1, 2] were published, many experiments have been designed to test it[3, 4, 5]. Generic format of Bell's inequality for continuous variable has been extensively studied[6, 7, 8, 9]. In addition to local variable theorem, studies on nonlocal variable theorem have been progressed both theoretically[10] and experimentally[11, 12, 13].

Presented here is an analysis of experiments testing Bell's inequality from a statistical point of view. First, it is revealed that each experiment testing Bell's inequality contains a 2-level statistical structure. This then leads to 2 important observations, which further give rise to the assumption of independence among EPR experiments[14]. It is then demonstrated that Bell's inequality can be derived with the necessary addition of this assumption. However, in such development, the upper bound for the CHSH-Bell observable is 4 for any and all individual experiment without locality consideration. Further reflection shows that our derivation is in agreement with Bell's theorem and the requirement of fair sampling can be dealt with in an alternative way. It is proposed that hypothesis testing should be used in analyzing the experimental data[15, 16].

The widely used formula in experimental data analysis on (1) is

$$\begin{aligned} &\hat{S}(a,a',b,b') \\ &= |\hat{E}(a,b)+\hat{E}(a,b')| + |\hat{E}(a',b)-\hat{E}(a',b')| < 2 \end{aligned} \tag{2}$$

where the calculation of $\hat{E}(a,b)$ from experimental data is

$$\hat{E}(a,b) = \frac{N_{ab}(+,+) - N_{ab}(+,-) - N_{ab}(-,+) + N_{ab}(-,-)}{N_{ab}(+,+) + N_{ab}(+,-) + N_{ab}(-,+) + N_{ab}(-,-)} \tag{3}$$

In the following discussion, S, E are the mean value in inequality (1), $\hat{S}, \hat{E}$ are the corresponding value computed from individual experimental outcome. As to be shown, $\hat{S}$ and S may be very different from each other. For clarity, S will be called CHSH-Bell variable(Bell variable for short), $\hat{S}$ will be called empirical CHSH-Bell observable(Bell observable for short).

Experiments testing Bell's inequality actually deal with two levels of experiments, the EPR experiment and the Bell experiment. An EPR experiment is a single measurement of the correlation between a pair of correlated particles, a Bell experiment includes four sets of EPR experiments, with each set of EPR experiments corresponding to one detector configuration, or equivalently, one term in (1). So, a Bell experiment is an ensemble of EPR experiments. This 2-level experiment hierarch leads to a 2-level statistical structure.

The 2-level statistical structure reflects the 2 parts in a Bell experiment, a) setup/run a single EPR experiment, take measurement and record the data, b) repeat a) and count the EPR experimental outcome to get the Bell observable via inequality (2) for testing against Bell's inequality. Typically part a) involves Quantum mechanical measurement that applies operators to the wave function and has been extensively studied[17, 18]. However, the counting in part b) has not drawn much attention. The present work will treat this counting issue. In general, EPR measurement in a) requires an experiment taking place in two regions of spacetime that are not causally related, but such requirement does not apply to the counting in b) and can not be enforced.

The 2-level statistical structure describes the relationship between random variables (experiments), which is common in statistical data analysis. A similar 2-level statistical structure can be found in a coin tossing experiment, where Bernoulli random variable (experiment) and Binomial random variable (experiment) are involved. A single coin toss gives a single Bernoulli random variable (experiment). If one counts the number N of head in M tosses, then N is a Binomial random variable (experiment). So a Binomial random variable (experiment) is an ensemble of Bernoulli random variable (experiment).

This 2-level statistical structure can lead to some important consequences. The first one (referred to as OB1 hereafter) is that the CHSH-Bell observable $\hat{S}$ computed from experimental data is actually a random variable. In general, the outcome of an EPR experiment is recorded as a random variable taking values of {+1, -1}. $N_{ab}(+,+), N_{ab}(+,-), N_{ab}(-,+), N_{ab}(-,-)$ in (3) are counts of EPR experiments, so they are all random variables. (Just as a counting of Bernoulli random variable gives a Binomial random variable.) Thus $\hat{E}(a,b)$ is a random variable. Therefore, the CHSH-Bell observable $\hat{S}$ is also a random variable. The value $\hat{S}$ computed from empirical data is a sample statistic.

Bell variable S, on the other hand, is an ensemble average, a constant when the measurement setup is fixed. In statistical analysis, S is a population parameter, which can be estimated from experimental outcome, but can not be measured/observed/computed directly from experimental data. This is why S should be called Bell variable, while $\hat{S}$ should be called empirical Bell observable. An important distinction between S and $\hat{S}$ is that S is a fixed value determined by

Table 1. Comparison of $\hat{S}$ and S

	$\hat{S}$	S
1	empirical sample average a statistical random variable	population (ensemble) average a fixed value
2	< 4	< 2 (by local HVT), $< 2\sqrt{2}$ (by QM)
3	can be directly computed from empirical data	can be estimated from experimental data, but can not be directly derived from experimental data
4	is a parametric family of random variable whose probability density function is determined by detector configuration a, a', b, b'	a single value uniquely determined by detector configuration a, a', b, b'

a, a', b, b' while $\hat{S}$ is a random variable whose probability density function is determined by a, a', b, b'. For convenience, Table 1 is a comparison of $\hat{S}$ and S. It turns out that S is the ensemble average of $\hat{S}$. Further clarification on the upperbound of S and $\hat{S}$ will be given later.

Another observation (referred to as OB2 hereafter) is that each EPR experiment can exist by itself and is not necessarily bound to any specific Bell experiment. Here is an analogue, in the coin tossing setup, a single tossing, a Bernoulli experiment, can be included in any counting Binomial random variable as long as the underlying probability is the same. Similarly, an EPR experiment can be included in any Bell experiment as long as the measuring detector configuration is consistent. On the other hand, a Bell experiment can remove any of its constituent EPR experiment by not counting it. (Due to detector efficiency, many EPR experiments are not counted.)

Since there is no specific association between designated EPR and Bell experiment, one can arrive at the assumption that EPR experiment outcomes within a Bell experiment are independent of each other. From this assumption, it will be shown that the Bell variable S in CHSH-Bell's inequality is an average of the empirical Bell observable $\hat{S}$.

To facilitate the discussion, here are some definitions that will be used later. We denote by A, B the detectors at the two sides of an EPR experiment, a, a' the two configuration of detector A, b, b' the two configuration of detector B. In a single EPR experiment, $A(\omega_A, \lambda)$ is the outcome at detector A in configuration of ω_A, $B(\omega_B, \lambda)$ is the outcome at detector B in configuration of ω_B, $A(\omega_A, \lambda)$ and $B(\omega_B, \lambda)$ take values of -1 and 1, λ is a hidden random variable with probability distribution $\rho(\lambda)$. $\lambda_{ab,i}$ is the hidden random variable in the i-th EPR experiment when the detector A, B are in configuration of (a, b). It is worth noting that the hidden random variable is directly associated to a single EPR experiment. Then

$$\begin{aligned}
&\sum_{i=1}^{N_{ab}} A(a, \lambda_{ab,i}) B(b, \lambda_{ab,i}) \\
&= N_{ab}(+,+) - N_{ab}(+,-) - N_{ab}(-,+) + N_{ab}(-,-) \\
&N_{ab} = N_{ab}(+,+) + N_{ab}(+,-) + N_{ab}(-,+) + N_{ab}(-,-)
\end{aligned} \quad (4)$$

Similiar equations can be derived for detector configurations (a, b'), (a', b), (a', b'), from

(2) and (3)

$$\hat{S}(a,a',b,b') = \frac{1}{N_{ab}} \sum_{i=1}^{N_{ab}} A(a,\lambda_{ab,i}) B(b,\lambda_{ab,i})$$
$$+ \frac{1}{N_{ab'}} \sum_{j=1}^{N_{ab'}} A(a,\lambda_{ab',j}) B(b',\lambda_{ab',j})$$
$$+ \frac{1}{N_{a'b}} \sum_{k=1}^{N_{a'b}} A(a',\lambda_{a'b,k}) B(b,\lambda_{a'b,k})$$
$$- \frac{1}{N_{a'b'}} \sum_{l=1}^{N_{a'b'}} A(a',\lambda_{a'b',l}) B(b',\lambda_{a'b',l}) \tag{5}$$

Each sum in equation (5) corresponds to a unique set of EPR experiments. Since EPR experiments are independent of each other, $\lambda_{ab,i}$, $\lambda_{ab',j}$, $\lambda_{a'b,k}$, $\lambda_{a'b',l}$ are independent random variables with the same probability distribution function $\rho(\lambda)$, rather than identical. Thus their joint distribution is $\rho = \prod_{i=1}^{N_{ab}} \prod_{j=1}^{N_{ab'}} \prod_{k=1}^{N_{a'b}} \prod_{l=1}^{N_{a'b'}} \rho(\lambda_{ab,i})\rho(\lambda_{ab',j})\rho(\lambda_{a'b,k})\rho(\lambda_{a'b',l})$, therefore the average of $\hat{S}$ is

$$\int \hat{S}(a,a',b,b')\rho d\lambda_{ab,1} \cdot\cdot d\lambda_{ab,N_{ab}} d\lambda_{ab',1} \cdot\cdot d\lambda_{ab',N_{ab'}}$$
$$d\lambda_{a'b,1} \cdot\cdot d\lambda_{a'b,N_{a'b}} d\lambda_{a'b',1} \cdot\cdot d\lambda_{a'b',N_{a'b'}}$$
$$= \int \rho(\lambda)[A(a,\lambda)B(b,\lambda) + A(a,\lambda)B(b',\lambda) + A(a',\lambda)B(b,\lambda)$$
$$- A(a',\lambda)B(b',\lambda)]d\lambda$$
$$= S(a,a',b,b') \tag{6}$$

here we used the following fact

$$\int \rho(\lambda_{ab,i}) d\lambda_{ab,i} = \int \rho(\lambda_{ab',j}) d\lambda_{ab',j}$$
$$= \int \rho(\lambda_{a'b,k}) d\lambda_{a'b,k} = \int \rho(\lambda_{a'b',l}) d\lambda_{a'b',l} = 1$$
$$\int \rho(\lambda_{ab,i}) A(a,\lambda_{ab,i}) B(b,\lambda_{ab,i}) d\lambda_{ab,i}$$
$$= \int \rho(\lambda) A(a,\lambda) B(b,\lambda) d\lambda$$
$$\int \rho(\lambda_{ab',j}) A(a,\lambda_{ab',j}) B(b',\lambda_{ab',j}) d\lambda_{ab',j}$$
$$= \int \rho(\lambda) A(a,\lambda) B(b',\lambda) d\lambda$$
$$\int \rho(\lambda_{a'b,k}) A(a',\lambda_{a'b,k}) B(b,\lambda_{a'b,k}) d\lambda_{a'b,k}$$
$$= \int \rho(\lambda) A(a',\lambda) B(b,\lambda) d\lambda$$
$$\int \rho(\lambda_{a'b',l}) A(a',\lambda_{a'b',l}) B(b',\lambda_{a'b',l}) d\lambda_{a'b',l}$$

Table 2. Experimental data set with a single sample for each configuration of detection apparatus A, B.

	A	B	$N(+,+)$	$N(+,-)$	$N(-,+)$	$N(-,-)$
1	a	b	1	0	0	0
2	a	b'	1	0	0	0
3	a'	b	1	0	0	0
4	a'	b'	0	1	0	0

$$= \int \rho(\lambda) A(a', \lambda) B(b', \lambda) d\lambda \tag{7}$$

Set

$$\begin{aligned} s(\lambda, a, a', b, b') = A(a, \lambda)B(b, \lambda) + A(a, \lambda)B(b', \lambda) \\ + A(a', \lambda)B(b, \lambda) - A(a', \lambda)B(b', \lambda) \end{aligned} \tag{8}$$

since $A(\omega_A, \lambda)$ and $B(\omega_B, \lambda)$ take values of -1 and 1, $s = -2$ or $s = 2$ for every detector configuration with arbitrary λ. Therefore we get the inequality (1) from (6).

Unlike other ways of directly putting the value s of equation (8) into the integral, our derivation works through equation (5) and (6), which are just manifestations of the independence assumption. Although our method gives the same bound as CHSH-Bell's inequality, equations (5) and (6) demonstrate the difference between the CHSH-Bell observable $\hat{S}$ and the CHSH-Bell variable S. This difference is common in statistical models, where a random variable usually takes value within a range while its mean takes a unique value. The inequality applies only to the CHSH-Bell variable S, rather than the CHSH-Bell observable $\hat{S}$. In fact the upper bound for CHSH-Bell observable $\hat{S}$ is 4, the set of experimental data shown in table 2 is illustrative.

Simple computation with equation (3) shows that

$$\begin{aligned} &\hat{E}(a, b) = 1 \\ &\hat{E}(a, b') = 1 \\ &\hat{E}(a', b) = 1 \\ &\hat{E}(a', b') = -1 \end{aligned} \tag{9}$$

It immediately follows that

$$\hat{S}(a, a', b, b') = 4 \tag{10}$$

The $\hat{S}$ in equation (10) is not only greater than the upper limit set in inequality (1), but it also exceeds the upper bound of $2\sqrt{2}$ set by quantum theory[19].($2\sqrt{2}$ is also an upper bound of mean value but not bound of any and all individual outcome) Moreover, the development does not rely on any assumption of locality, nor is it in any relation to a specific configuration a, a', b, b' of the detectors. It is purely statistics.

Equation (10) is a direct consequence of (5). Due to the assumption of independence, $\hat{S}$ in equation (5) is the proper value for individual outcome, and $\lambda_{ab,i}, \lambda_{ab',j}, \lambda_{a'b,k}, \lambda_{a'b',l}$ in (5) are independent, thus (5) admits data shown in Table 2. On the contrary, s in equation (8) is derived from $\hat{S}$ by way of integration[20] in the process of averaging and is right for computation of the mean value, thus requires identical λ for each of the four terms and is incompatible with Table 2. This difference just shows that in a statistical model, an upper limit of mean value is distinct from the upper limit of any and all individual experiment outcome[21].

In addition to the 2-level statistical structure, here is more clarification on the independence assumption and equation (5). The only constraint on the four terms in the Bell's inequality (1) is that they are correlations for detectors configured at (a, b), (a', b), (a, b') and (a', b') respectively, Bell's inequality does not put any requirement on how the experimental data should be collected, in particular, it does not require the four sets of experiments to be conducted at the same place and same time. On the other hand, it is impossible to measure all relevant terms simultaneously in a single experiment[22]. As a matter of fact, the earth is spinning, and is orbiting around the sun, the sun is rotating within the milky way, so the four sets of EPR measurement taken in some lab on the earth do not actually happen at the same place. Given the above consideration, NO Bell experiment has ever collected all four sets of EPR experiment data at the same place and at the same time. In theory, one can collect EPR experimental data for detectors in configuration of (a, b) in a lab on earth and conduct similar EPR experiments on Mars to collect data for detectors in configuration of (a, b'), one can also run EPR experiments to collect data in configuration (a', b') some years after the EPR data in configuration (a, b) is first collected. To push it to extremity, one can even get EPR data in configuration (a, b) from a photon experiment and EPR data in configuration (a', b) from a massive particle experiment. These EPR experimental data can then be put together to form the Bell observable $\hat{S}$ to test against Bell's inequality the same way as if they were collected at the same place within a limited time span. Therefore, data collection experiments for each of the four terms in inequality (1) are independent.

The choice of $\lambda_{ab,i}, \lambda_{ab',j}, \lambda_{a'b,k}, \lambda_{a'b',l}$ in equation (5) can significantly change the scope of local theory under consideration and the way of Bell's inequality being tested. 2 scenarios: 1) $\lambda_{ab,i} = \lambda_{ab',j} = \lambda_{a'b,k} = \lambda_{a'b',l} = \lambda$, 2) independent $\lambda_{ab,i}, \lambda_{ab',j}, \lambda_{a'b,k}, \lambda_{a'b',l}$, are considered here and it is shown that the second scenario is necessary for all local theories to be included and tested. In the first scenario, if it were enforced $\lambda_{ab,i} = \lambda_{ab',j} = \lambda_{a'b,k} = \lambda_{a'b',l} = \lambda$, the upper bound of Bell observable $\hat{S}$ would be 2 for any and all individual Bell experiment and a single individual experiment with $\hat{S} > 2$ would be enough for the violation of the inequality. However, in this case, λ would be constant[23], the theories being tested would be in the space of hidden constant theory, rather than the space of hidden variable theory. In the second scenario, there is no restriction on $\lambda_{ab,i}, \lambda_{ab',j}, \lambda_{a'b,k}, \lambda_{a'b',l}$ so the whole space of hidden variable theory are included. But in this case, as shown in equation (10), the upper bound of Bell observable $\hat{S}$ is 4 and can not be violated in any individual experiment. The above 2 scenarios show that, 1) violation of Bell's inequality with individual experiment can only rule out local hidden constant theories. 2) Bell's theorem sets the upper bound for the statistical mean, rather than for any and all individual experiment. To include and test all local hidden variable theories, one must admit the independence of $\lambda_{ab,i}, \lambda_{ab',j}, \lambda_{a'b,k}, \lambda_{a'b',l}$ and work with the statistical mean (see equations (5)-(8)).

The validity of equation (10) is also necessary for violation of Bell's inequality. For the sake of this discussion, we denote by $D_{M,N,P,Q}$ the set of all Bell experiments that have M, N, P, Q EPR experiments for each of the four detector configuration (a, b), (a', b), (a, b'), and (a', b') respectively and $E_{M,N,P,Q}$ an element of $D_{M,N,P,Q}$ (e.g. $E_{M,N,P,Q}$ is an individual Bell experiment with M, N, P, Q EPR experiments for each of the four detector configuration (a, b), (a', b), (a, b'), and (a', b')). Then the data in Table 2 is associated to an individual experiment $E_{1,1,1,1}$. Given a Bell experiment $E_{M,N,P,Q}$, one can always pick $m < M, n < N, p < P, q < Q$ samples from

each of the four detector configuration (a,b), (a',b), (a,b'), and (a',b') and form an element $E_{m,n,p,q}$ in $D_{m,n,p,q}$, which will be called sub-Bell Experiment[24]. The Bell observable $\hat{S}$ for elements $E_{1,1,1,1}$ can only take values in $\{0,2,4\}$. In any experiment violating Bell's inequality, there must be sub-Bell experiment $E_{1,1,1,1}$ with $\hat{S}>2$, then the only choice is $\hat{S}=4$.

It should be emphasized that our derivation of (10) is in agreement with Bell's inequality. In fact, $\hat{S}$ in equation (10) is the empirical CHSH-Bell observable (a random variable) while S in (1) is the CHSH-Bell variable. After averaging, Our approach gives the same upper bound of 2. Using $\hat{S}$ to test the violation of Bell's inequality requires the issue of fair sampling to be addressed. The 2-level statistical structure brings an alternative way of dealing with the fair sampling assumption.

The current treatment of fair sampling is at the EPR experiment level. It is requiring fair sampling of the EPR experiments within a single Bell experiment (i.e. the sampling of +/-1 should be fair), and taking $\hat{S}$ as a faithful copy of S for test of Bell's inequality. But fair sampling should also be required at the Bell experiment level, that is, taking fair sampling of $\hat{S}$ within the interval from 0 to 4. In fact, the fair sampling treatment at the Bell experiment level is more desired. A similar case can be found in a coin tossing experiment, one can require fair sampling of the Bernoulli experiment so that a particular outcome of the ensemble Binomial random variable (experiment) $\hat{N}=N_{mean}$, but it does not change the fact that $\hat{N}$ is a Binomial random variable and can take any value in the range.

With the 2-level statistical structure, the fair sampling assumption can be dealt in an alternative way. Rather than requiring fair sampling within an individual measurement of $\hat{S}$ at the EPR experiment level, one could take multiple measurement of $\hat{S}$ and fair sampling at the Bell experiment level, then evaluate its probability distribution and draw conclusion therefore. Since $\hat{S}$ is a statistical random variable, spreading from 0 to 4, in an ideal experiment design, data over the whole range should be obtained so that the probability distribution of $\hat{S}$ is faithful and meaningful conclusion can be drawn.

One obstacle is that S and $\hat{S}$ are quite different from each other, as shown in Table 1. The only available experimental data is the empirical Bell observable $\hat{S}$, which is a random variable whose range is from 0 to 4, whereas Bell's inequality deals with S and puts no direct restriction on every individual experimental outcome $\hat{S}$. In statistical inference, comparison of a random variable with a fixed value does not draw any conclusion directly. This is usually treated as a hypothesis testing problem[15, 16]. In specific, Bell's inequality (1) serves as the null hypothesis, $\hat{S}$ would be test statistic, and the outcome would be accepting or rejecting the hypothesis with some predefined significance level.

In summary, experiments testing CHSH-Bell's inequality are studied with a statistical point of view and a 2-level structure is revealed. All experimental implementations are only individual experiments, and the outcome is just the Bell observable $\hat{S}$, a random variable. Bell's inequality states that the Bell variable S is less than 2 for any hidden variable theory (HVT), but NOT that $\hat{S}$ in every individual experiment outcome is less than 2 for any HVT. The upper bound of $\hat{S}$ for any and all individual outcome is 4 regardless of locality consideration, which is distinct from the upper bound of its mean value $S_{max}=2$. Therefore violation of Bell's inequality with these individual experiments should be statistically significant. The fact is that 1) the inequality deals with S but S can not be obtained from any experiment. 2) $\hat{S}$

can be acquired from experiment but it is not directly regulated by the inequality. So the solution is hypothesis testing. Fair sampling should also be done at the level of Bell experiment, that is, all outcome of $\hat{S}$ in the range from 0 to 4 should be taken into account.

Finally is a speculation that the local realism might be preserved given currently available experimental data. With the maximally entangled setup, rather than requiring fair sampling within a single Bell experiment, one could take a number of Bell's experiments and see how many outcomes lead to $\hat{S} > 2$ and $\hat{S} < 2$ respectively and test the significance for each case. One notable fact is that, violation of Bell's inequality is always a sophisticated experiment and only a small number of labs can achieve it. This seems implying that in a large number of similar experiments $\hat{S}$ actually failed to break the bound of 2. Therefore if hypothesis testing is applied to the analysis of the complete experimental data, there is a real chance of local realism being accepted, even if all loopholes can be closed in a single experiment.

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[20] $\int f(\lambda_{ab,i})d\lambda_{ab,i} = \int f(\lambda_{ab',j})d\lambda_{ab',j} = \int f(\lambda_{a'b,k})d\lambda_{a'b,k} = \int f(\lambda_{a'b',l})d\lambda_{a'b',l} = \int f(\lambda)d\lambda$, the difference in $\lambda_{ab,i}, \lambda_{ab',j}, \lambda_{a'b,k}, \lambda_{a'b',l}$ is removed after integration.
[21] In an experiment, one can toss a fair coin M times and count the number N of head. In theory, the mean $N_{mean} = M/2$. However, Binomial distribution shows that it is possible that some experiments produce outcome of $N = M > M/2$.
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[23] Consider two individual Bell experiments with detector configuration (a, b, a', b') and (a, b, a'', b'') respectively. Both experiments contain a common term $A(a, \lambda)B(b, \lambda)$. If we enforce the four sums of each of the two Bell experiment having the same λ, then all terms of both of the two Bell experiments have the same λ. It then follows that all individual experiments have the same λ.
[24] Due to detector efficiency, the collected data is picked by the detector in a subset of the whole experimental outcome, thus also a sub-Bell experiment.